The true magnitude of Absolute Totality

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June 2021

Abstract

In this paper we introduce the reader to mental constructions which far surpass anything present in modern analytic philosophy and provide arguments against the well-established consensus that the scope of proper classes adequately reflects the magnitude of absolute Everything. An argument is provided that some of the greater mentally accessible constructions described in the paper are too great (in terms of cardinality and lengths of orderings) to have a formal structural ontology and so are incompatible with metaphysical positions such as ontic structural realism which some philosophers don't view as severely limiting.

1 Defence

To enter the realm of the constructions described in the paper we must first defend its conceptual coherence.

We shall present a nonstandard interpretation of the Burali-Forti paradox.

1.1 The modern perspective

As a consequence of the developments in the foundations of mathematics, set theory is seen as grounding mathematics as a whole. Formalists' conclusion is that all mathematical structures and phenomena can always be treated as sets with relations defined over them. Platonists' conclusion is that the realm of abstract mathematical objects is one-sorted — every abstract mathematical object is ontologically a set. Accordingly, proper-classes are seen as ethereal phenomena which do not bear actual existence as abstract individuals but instead lie at the very limit of the realm of abstract objects as a whole.

A major role in the formation of the view that proper classes as the untranscendible, or at least, not with philosophically fruitful results, was played by Georg Cantor's own views on the subject. Cantor linked the Absolute Infinite with God, and believed it satisfied the full reflection principle: every property of the Absolute Infinite is also held by some smaller object. Cantor's insight is so crucial that it still motivates the biggest subject in set theory — large cardinals. The stronger a large cardinal is the closer properties to full reflection principle it bears.

Burali-Forti paradox is then seen as reaffirming the conclusion that to speak of abstract objects is to speak of sets and that to speak of quantification over absolutely every abstract object (or even absolutely everything if one accepts ontic structural realism) is to quantify over the proper class V. Extending the idea, the consensus is that proper classes and higher-order collections are to be seen as nothing but byproducts of second-order and higher order quantification over V.

1.2 Burali-Forti paradox and its relatives

To present a different perspective on the Burali-Forti paradox we must first deconstruct it. Consider the following family of paradoxes:

FINITE WELL-ORDERINGS

Assume all well-orderings are finite (1)

Consider the well-ordering of all finite well-orderings, notate it as Ω (2)

Then, since all well-orderings are finite, Ω itself is a finite well-ordering and must be a proper initial segment of the well-ordering of all finite well-orderings

It follows that Ω is a proper initial segment of itself

For any finite well-ordering, its successor well-ordering is also finite

It follows that $\Omega + 1 < \Omega$ contradiction.

As a consequence, we are forced to abandon one of the two assumptions:

The assumption that all well-orderings are finite. (1)
The assumption that the well-ordering of all finite well-ordering exists. (2)

Set theorists, mathematicians and most modern analytical philosophers merrily abandon the assumption (1) and accept the realm of transfinite.

TYPES OF WELL-ORDERINGS IN GENERAL

Assume all well-orderings are computable/countable/less than 27th Woodin cardinal

Consider the well-ordering of all ——— well-orderings, notate it as Ω

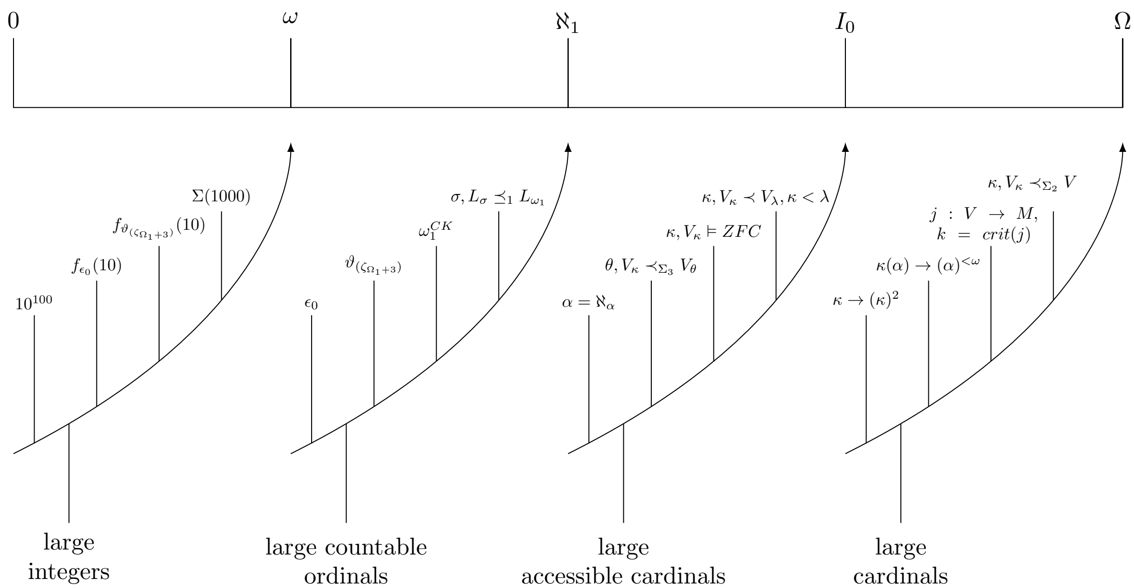
Then, since all well-orderings are ———, Ω itself is a ——— well-ordering and must be a proper initial segment of the well-ordering of all ——— well-orderings.

It follows that Ω is a proper initial segment of itself

For any ——— well-ordering, its successor well-ordering is also ———
It follows that $\Omega + 1 < \Omega$ contradiction.

As seen from this generalized argument the analogues of Burali-Forti paradox have exact same structure as the original paradox, and the analogues are valid as long as the supremum of all existing well-orderings is assumed to be a limit well-ordering. Set theorists have no issue breaking past this limit simply by considering more general kinds of well-orderings then the limit well-ordering admits.

However the original Burali-Forti paradox stands out as a barrier harder to break



Well-orderings can be grouped in several ontologically significant categories.

- Finite ordinals.
- Infinite ordinals. ω introduces the realm of transfinite structures with properties completely unseen in the realm of the finite. Ontological finitists already view infinite ordinals as nothing but a byproduct of mathematical "symbolic games".
- Uncomputable ordinals. Ordinals past ω_1^{CK} introduce intricacies of admissible ordinals unseen in the computable realm. Those who subscribe to metaphysical analogues of Church–Turing–Deutsch principle view uncomputable ordinals similarly.
- Uncountable ordinals. Ordinals past ω_1 introduce previously unseen uncountability and its consequences.
- Large cardinals. Ordinals past the first worldly cardinal introduce the vastly rich world of large cardinals.

However well-orderings past Ω introduce seemingly nothing. There is seemingly nothing special, no conceptual gain from considering $\Omega+1$ or $\Omega \cdot 2$ or Ω^+ except the technical and ontological difficulties behind such well-orderings. We are tempted to accept the conclusion that such well-orderings are not really ontologically grounded, are rather problematic while being completely unfruitful. After all, treating every abstract mathematical object ontologically as a set proves to be wildly successful and fruitful.

In fact, proper-class sized well-orderings are so similar to set-sized well-orderings that they are naturally elementarily equivalent to much smaller, large cardinal-sized well-orderings. Reductions of the considered well-orderings virtually never end up exceeding Mahlo cardinals. Those who considered the proper class -sized and greater well-orderings are then invited to study the large cardinal hierarchy, which is really where all of the fruitfulness lies. Suddenly, all hopes of finding something remarkable or ontologically meaningful about well-orderings past Ω are lost.

1.3 Platonists and Formalists

To find ontological meaning and non-triviality of proper class -sized and greater well-orderings we shall consider the following analogy:

Countable models $\mathcal{M}$ of ZFC fail to satisfy external fullness of the powerset of $\mathbb{N}$ which is something Platonists find philosophically unsatisfying about such models. Formalists, on the other hand, aren't bothered by such models at all as the metatheoretically-definable subsets are the only kinds of subsets truly relevant to the formal study of set theory. As a consequence ω_1, to formalists, is nothing but a useful fiction and they have no need of ontological commitment to any ordinals greater than very large countable ordinals such that for all practical purposes they are indiscernible from uncountable ordinals.

Analogously, irreducibility of Ω to sets is viewed by set-theoretical Platonists as uncountability of ω_1 is seen by formalists - It doesn't bear ontological existence, since $(\Omega, <)$ is viewed by set theoretical Platonists not as an individual object but rather a byproduct of considering the realm of abstract objects in full generality. Set-theoretical Platonists, understandably, picture the realm of all abstract objects as proper-class sized, after all, the idea that every abstract mathematical object is a set is wildly successful. Extending this idea, Platonists picture absolute Everything as proper-class sized as other kinds of objects, beside abstract mathematical don't seem to significantly add up to the total amount. This position is ontologically comfortable for set-theoretical Platonists and is virtually never put to doubt by other philosophers. It is simply unlikely for them to seriously consider large cardinals let alone well-orderings past Ω in the setting of ontology in the first place, and even if they do, at the first glance Burali-Forti paradox seems to be an argument compelling enough against the impossibility of absolute Everything being any greater than Ω.

1.4 The alternative perspective

Ω can and should be appreciated as the supremum of all well-orderings amenable to a single logical sort, in case of Ω the sort involved is "set". Being the supremum, Ω is by an argument, obviously similar to Burali-Forti paradox, not amenable to a single logical sort. We shall notate the property of Ω of not being amenable to single logical sort as "1-sort irreducibility". [accordingly, being amenable to a single logical sort shall be notated as 1-sort reducibility].[‡]

Well-orderings α which are a β-sort irreducible shall be notated as $\alpha_{\dagger\beta}$.
The least well-ordering which is β-sort irreducible shall be notated as $\omega_{\dagger\beta}$.

1-sort irreducibility of $\omega_{\dagger 1}$ should be seen exactly in the same light as uncountability of ω_1. Just the same way the conception of ω_1 doesn't signify that we should stop and reaffirm that all mathematical objects are countable and that ω_1 bears no actual platonic existence, we should instead view the conception of 1-sort irreducibility of $\omega_{\dagger 1}$ as the first step towards something great, much greater in scale then the all of the previously considered conceptions. $\omega_{\dagger 1}$ instantly makes us wonder what kind of beasts $\omega_{\dagger 2}$ and $\omega_{\dagger\Omega}$ are, just like ω_1 makes us instantly wonder what kind of beasts ω_2 and ω_ω are. Usually many-sorted theories of sets and classes seem jumbled and uninspiring, however once we look at Ω from the perspective of 1-sort irreducibility we gain a newfound appreciation for such treatments.

1.5 Scales of α-sort irreducibility

Let us realize the massive scale of the leap from the supremum of all alpha-irreducible well-orders to the supremum of all alpha+1-irreducible well-orders.

α-sort irreducibility, unlike superclass-level isn't exhausted at the "cardinal successor", for lack of a better term, of a collection we started with. A well-ordering too great to be proper class-sized is a mere proper superclass and acts in about every way as "the next initial ordinal past Ω", while well-orderings too great to be 2-sort reducible are much *much* greater.

1.6 Standard models of a special type of Ω-sized theories

Consider a one-sorted theory $\mathfrak{T}$ with Ω-many constants c_α for each ordinal α and Ω-many axioms stating that the constants are pairwise non-equal. Let c_Ω be an additional constant stated to be pairwise non-equal to all c_α for ordinal α via Ω additional axioms. Assume ZFC_2 is a fragment of $\mathfrak{T}$. Let $\phi(x)$ be the predicate of being an initial ordinal. Let $\mathfrak{T}$ contain axioms of the form $\phi(c)$, for each

[‡]Further in the paper we shall present an argument why well-orderings of the form $\omega_{\dagger\beta}$ should be seen as β-th instances of the "full" reflection principle envisioned by Georg Cantor.

constant of the theory. Finally, let $\mathfrak{T}$ contain axioms of the form $\psi(c_\alpha, c_\Omega)$, for each ordinal α, where $\psi(\beta, \gamma)$ is a predicate "β is a lesser ordinal then γ".

Since $\mathfrak{T}$ is one-sorted it is sort-nonstandard, meaning all of its models are externally greater than than 1-sort reducible by size. If $\mathfrak{T}$ is viewed as object theory, it is externally, in other words, meta-theoretically unsound with respect to the amount of sorts in its models. If $\mathfrak{T}$ is viewed as metatheory, it is externally, in other words, meta-meta-theoretically unsound accordingly.

At cost of being sort-nonstandard, $\mathfrak{T}$ is able to axiomatize 1-sort irreducible collections greater in cardinality than Ω **concretely**, meaning without the circularity of using greater than Ω-many axioms or constants. There is no necessity to stop at $\mathfrak{T}$, one could easily construct theories with 2 cardinals above Ω, ω cardinals above Ω, a 53-huge cardinal above Ω, etc., modulo trivial adjustments to the construction of $\mathfrak{T}$.

However, the supremum of the size of least models of concrete theories is a clear limit. Let $\omega_{\dagger 1}^{\mathfrak{M}}$ be the notation of the supremum. It is notable that $\mathrm{cof}(\omega_{\dagger 1}^{\mathfrak{M}}) = \Omega^+$. One could extend the construction to one-sorted theories which are not concrete or construct hierarchies of "concreteness degrees" akin to the hierarchy of admissible ordinals.

Such hierarchies get increasingly rich and layered, and thus reflection properties should eventually emerge. Initial segments α which satisfy such properties are the concreteness degree analogues of large cardinals and the limit of α which admit such reflection principles should be analogous to Ω. The property which grounds full reflection at Ω is its set-irreducibility, and analogously, the property which grounds full reflection at the limit is 2-sort irreducibility.

1.7 Nested limit-Universe construction

Consider a two-sorted theory of sets, one for "small" sets and one for "large" sets. Small sets are elements of $V_{<\kappa}$, κ being an instance of an inaccessible cardinal, and all sets are "large".

If axioms of ZFC hold for large sets and κ is the least inaccessible cardinal we get the simplest case of a doubly nested Grothendieck-Zermelo universe. Instead, we may have considered κ to be closed under taking the Mahlo operation or taking the next weakly compact, or the next 53-huge, or some larger cardinal each time building an increasingly larger initial Grothendieck-Zermelo universe, however, without any additional boost on how much further the top-layer Grothendieck-Zermelo universe extends. To extend the nested universe on both ends should have considered that the layer of large sets is also closed under taking Mahlo operation, weakly compact, 53-huge, etc., as long existence of proper-class amount of such cardinals is consistent. Extending our theory of a 2-nested Grothendieck-Zermelo universe on both layers with large cardinals we get increasingly rich structure.

Completing the idea, we may, philosophically, consider the limit of such extensions.* We would get a 2-nested Grothendieck-Zermelo universe whose bottom layer is already Ω-tall and whose outer layer extends to something much greater than mere next inaccessible, Mahlo or the next 53-huge. The height of the second layer would be $\omega_{\dagger 1}$ — the analogue of Ω for 1-sort irreducible collections. It would be the supremum of all 2-sort reducible and, as a consequence, be the least 2-sort irreducible well-ordering.

If we considered the analogue of the limit-extension of a 3-nested Grothendieck-Zermelo universe we would get $\omega_{\dagger 2}$ — the analogue of Ω for 2-sort reducible collections as the height of the overall intended interpretation.

We may then consider the similar limit extension for w-nested Grothendieck-Zermelo universes, I_0-nested Grothendieck-Zermelo universe and Ω-nested Grothendieck-Zermelo universes. We may also consider more intricate structures such as limit extensions of Ω+1-nested Grothendieck-Zermelo universes or limit-extensions of κ-nested Grothendieck-Zermelo universes, where κ is the least 2-sort irreducible well-ordering. If we wish the height of the intended interpretation to be the "12th extendible cardinal past the supremum of 2-sort reducible well-orders" we could cook up the appropriate limit-extension construction. In this particular example case we would start with a 3-sorted theory of a 3-nested Grothendieck-Zermelo universe. Postulate the full reflection of the small and moderate universes and postulate the existence of sets up to the next 12-extendible on the top layer.

*Mere closure under large cardinals is far, far from enough due to concreteness degrees, yet it is still possible to mentally address such proper completion in the same manner Platonists mentally address Th($\mathbb{N}$) or $(\Omega, <)$.

In fact, there is no need to stop at α-sort irreducibility for finite α. We can conceptualize and appreciate the colossal well-orders which are ω-sort irreducible, Ω-sort irreducible, $\varsigma = \varsigma_{\dagger\varsigma}$ etc.

1.8 Relation between $\mathcal{P}(\alpha)$ and the α-sort successor operation

It may seem straightforward that the collection of all α-sort irreducible well-orderings is closed under the powercollection operation, for each well-ordering α, however, without explicitly assuming this principle as an axiom it is consistent that it is not the case. Under the perspective of α-sort irreducibility the axiom of powerset in ZFC, states that powerset operation preserves 1-reducibility, we shall propose a similar, more general axiom.

Global Powercollection †-Idempotence. For all well-orderings α,

$$\dagger(\mathcal{P}(\alpha)) = \dagger(\alpha)$$

Without the presence of **GPI**, well-orderings α such that $\dagger(\mathcal{P}(\alpha)) = \dagger(\alpha)$ shall be notated as †-idempotent.

A slightly weaker axiom shall also be considered.

Moderate Powercollection †-Idempotence. For all well-orderings α, β

$$(\dagger(\mathcal{P}(\alpha)) = \dagger(\alpha)) \longrightarrow ((\dagger(\beta) = \dagger(\alpha)) \longleftrightarrow (\dagger(\mathcal{P}(\beta)) = \dagger(\alpha))$$

While **GPI** might be false any set theoretical platonist would agree that **MPI** holds at least in some structures.

Axioms such as **GPI** and **MPI** may or may not hold in initial segments of the hierarchy of all α-sort irreducible well-orderings ordered under order-preserving embeddings. **GPI** certainly holds in V and its violation in $\omega_{\dagger 1}$ would contradict the principle of *minimal cumulative structural violation.*

Under the same reasoning, it should be the case that $\omega_{\dagger 2}$, $\omega_{\dagger 3}$, etc. $\models$ **GPI**. On the other hand violation of **GPI** is less structurally severe than violation of well-orderability so the least well-ordering α such that $\alpha \not\models$ **GPI** must exist and should in some way contradict the premises of the proof above. This observation hints towards the hierarchy of instances of violations of **GPI** and principles similar to it. Perhaps some kinds of such instances play the role of "large cardinal" properties in the realm of α-sort irreducble well-orderings.

1.9 Large Cardinals beyond Ω

The notion of α-sort irreducibility provides us with a completely new perspective on the well-orderings past Ω and allows us to create a hierarchy of increasing scope and complexity akin to the hierarchy of large cardinals or large countable ordinals.

Definition 1.1 (α-sort fixed point)**.** κ is α-sort-inaccessible iff κ is a κ-sort irreducible

Definition 1.2 (α-sort inaccessible)**.** κ is α-sort-inaccessible iff κ is a κ-sort irreducible and $cof(\kappa) = \kappa$

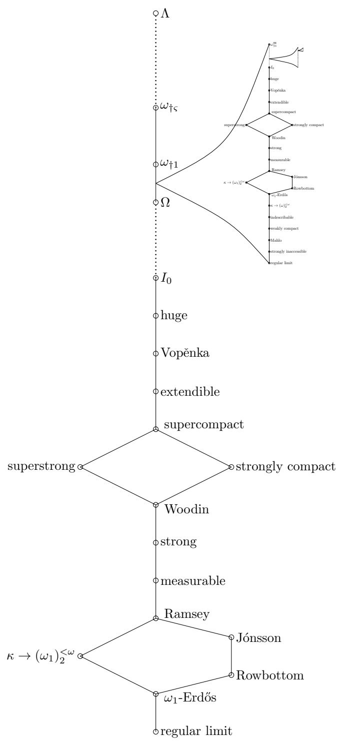
Definition 1.3 (α-sort Mahlo)**.** κ is α-sort Mahlo iff κ is κ-sort irreducible, $cof(\kappa) = \kappa$ any normal function f : $\kappa \to \kappa$ has a α-sort inaccessible fixed point.

Most variations of fixed points, inaccessibles and Mahlo cardinals translate straightforwardly into α-sort irreducible analogues.

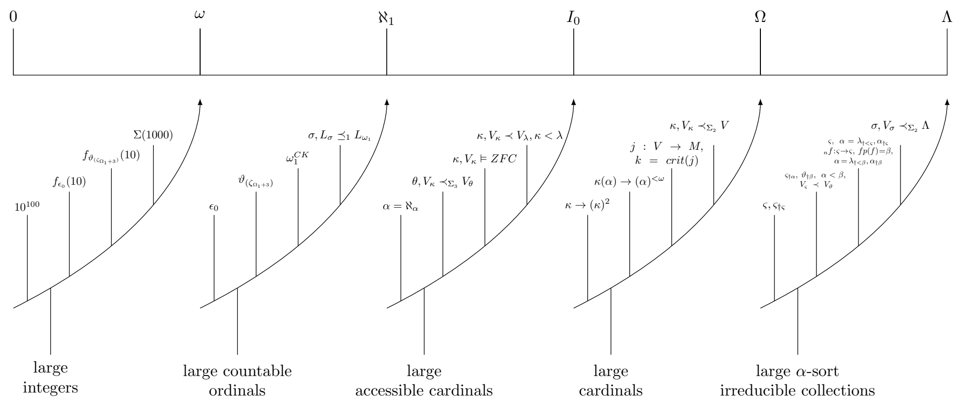
1.10 Nested Large Cardinal hierarchies

Hierarchies of increasingly reflective initial segments of the hierarchy of all well-orderings become increasingly deeply nested in rapid succession. Figuratively speaking, the hierarchy of concreteness degrees relates to the hierarchy of large α-sort irreducible well-orderings the same way the hierarchy of recursively large ordinals relates to the hierarchy of large cardinals.

$$\text{rec. large }\kappa \xmapsto{\omega_1 \mapsto \Omega} \text{large }\kappa \;\cong\; \omega_{\dagger\alpha}^{\mathfrak{M}} \xmapsto{\omega_{\dagger 1} \mapsto \Lambda} \text{large }\omega_{\dagger\alpha}$$



Recognizing ω_1 as "recursively Ord" and $\omega_{\dagger 1}$ as "concretely Λ" we can now fully envision the mathematical non-trivially in well-orderings past Ω. It is now particularly clear that, while Ω contains only a single copy of the large cardinal hierarchy, most well-orderings past Ω contain colossal amounts of similar such hierarchies simultaneously. As a consequence, one may study their ordering, consider special kinds of well-orderings which form initial segments of particular hierarchies or create and study more intricate notions related to the hierarchies. It is unclear how deeply nested the hierarchies can get and establishing the approximate limit is a also possible direction for future research.



2 Going much further

2.1 Deconstructing the concept of well-orderings

Having made a huge leap into the realm of α-sort irreducible collections we may still go forward and make a colossal leap.

What if we consider the limit of all these α-sort irreducible well-orderings? The limit, in Burali-Forti fashion, would be something which is the supremum of the kind of well-orderings we are considering.

It might be not initially clear that α-sort irreducibility was our last resort for extending the concept of well-ordering past Ω. However it will be clear if we call to mind the definition of a well-order as a concept.

A well-order is precisely a well-founded linear order.

Note that the cardinality or size issues of the well-ordering do not cause contradictions as long as the sizes are well-orderable. For this reason a well-ordering of an amorphous set is inherently contradictory but well-ordering of a proper class or a greater collection is not.

Now, note that the concept of well-order κ which is α-sort irreducible is never inherently contradictory as long as κ and α are both well-orderable. If α is limit well-ordering then it is naturally cofinal in κ, otherwise it is embeddable as a well-order in κ (consider the initial well-orders which are $<\alpha$-sort irreducible, these initial segments of κ would be elements of the well-ordering of α under the same well-ordering inherited from κ). For this reason, once we run out of α-sort irreducible well-orderings we run out of well-orderings in general, regardless of their size.

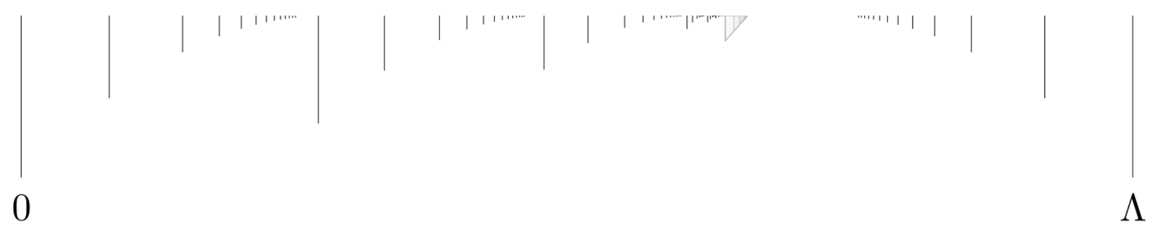
The only way to extend the hierarchy is to cast aside some of the essential non size-related properties of well-orderings. To compromise linearity of the ordering is clearly too drastic of a difference, our hierarchy was in many ways about length, so it is vital to the hierarchy. To compromise classical logic for paraconsistent is more reasonable but still is too hasty. We should compromise well-orderability minimally. Clearly, reconsidering the whole background logic may not be the most minimal compromise yet.

We shall turn our attention to well-foundedness, we shall consider a particular family properties which violate it. The properties form a very natural hierarchy by severity of the violation.[‡]

2.2 ω-almost-wellorderings

Define an order to be ω-almost-wellordered iff it is linear and contains no strictly decreasing chains of length ω^2, the motivation for such a definition will be clear further down the text.

In fact, for things to remain consistent and relatively similar to previous endeavours we need to drop another important property of well-orders. Supremum can no can longer always be the union of previous orders, as existence of such unions in some special cases is inconsistent. To work around that we declare the union to be slightly longer, in a precise sense, and restrict ourselves from taking the initial segment at exactly the union as a valid sub-collection.



Definition 2.1 (Λ)**.** Λ is the supremum of all well-orderings regardless of their collection types.

Λ, has no well-ordered initial segment that is equal to exactly the union of all well-orderings. Λ also contains "technical elements" at the end of it ordered as reverse ω. This is to cause slight ill-foundedness to avoid Burali-Forti paradox. A notable property of Λ is that any its initial segment is either isomorphic to itself or is well-ordered. Note that any decreasing sequence in Λ is not longer than $\omega + n$, for some finite ordinal n, so Λ is ω-almost-wellordered We can perform usual ordinal operations on Λ while retaining the essential properties of ordinal arithmetic as well as retaining ω-almost-wellorderability. Ordinal successor needs to be slightly redefined.

Definition 2.2 (The 3 fundamental types of ω-almost-wellorderings)**.**

1. Successor: λ which satisfy $\alpha + 1 = \lambda$ for some ω-almost-wellordering α.
2. Limit: λ such that $\forall \alpha < \lambda$, $\exists \beta$ such that $\alpha < \beta < \lambda$, α, β are ω-almost-wellorderings.
3. ω^*-point (special case of limit): λ such that $\exists$ technical b $\in \lambda$ not bounded by any non-technical a $\in \lambda$.

Definition 2.3 (Ordinal successor of an ω-almost-wellordering)**.** Ordinal successor of λ is the shortest (up to isomorphism) ω-almost-wellordered order extending λ not order isomorphic to λ.

[‡]In a precise sense, the property of self-containment is the supremum of the hierarchy.

Remark. *The definition prevents reverse-ω-points from being their own successors and prevents their successors from being limit.*

Definition 2.4 (Ordinal sum $\alpha + \beta$ of ω-almost-wellorderings and well-orderings)**.** Ordinal sum for non-ω^*-point α and arbitrary β is the lexicographic union $\alpha \cup \beta$; for ω^*-point α and arbitrary β is the lexicographic union $\alpha \cup \omega \cup \beta$.

Remark. *This definition prevent an analogous pathology.*

Definition 2.5 (Ordinal product $\alpha \cdot \beta$ of ω-almost-wellorderings and well-orderings)**.** Ordinal product of α and β is the lexicographic product $\alpha \times \beta$.

Definition 2.6 (Ordinal exponentiation α^β of ω-almost-wellorderings and well-orderings)**.** Ordinal exponentiation of α and β is the lexicographic order on the Cartesian power α^β.

Definition 2.7 (Principle of preservation of technical elements)**.** Technical elements are defined to be idempotent under left multiplication and left exponentiation for all ω-almost-wellorderings $\geq \Lambda$.

Remark. *This is to prevent pathological behavior of the operations at technical elements and make the hierarchy of ω-almost-wellorderings a more direct generalization of the hierarchy of ordinals.*

Theorem 1. *Λ+1 and Λ+α, where α is any well-order is ω-almost-wellordered; $\Lambda \cdot 2$ is ω-almost-wellordered.*

Proof. Throughout the orders at most 2 copies of ω^* are present, the rest of the order is well-ordered, meaning maximally long decreasing chains are of length $\omega \cdot 2 + n$, for finite ordinals n. □

Theorem 2. *$\Lambda \cdot \alpha$, where α is any well-order is ω-almost-wellordered.*

Proof. Throughout the order the copies of ω^* are ordered as α, which implies that, in the decreasing chains of the resulting order, elements of only finitely many copies of ω^* can be present [otherwise well-orderability of the order of copies of ω^* would be contradicted]. □

Theorem 3. *$\Lambda \cdot \lambda$ is an ω^*-point iff λ is any successor or ω^*-point.*

Proof. If λ is a successor then $\Lambda \cdot \lambda$ ends in a copy of λ which ends with technical elements ordered as reverse ω, If λ is an ω^*-point then $\Lambda \cdot \lambda$ by preservation of technical elements $\Lambda \cdot \lambda$ ends with technical elements ordered as reverse ω. If λ is limit then the resulting order contains neither the last copy of Λ nor preserved technical elements and hence, is a non ω^*-point limit. □

Theorem 4. *Λ^α, under ordinal exponentiation, where α is an arbitrarily well-order is ω-almost-wellordered.*

Proof. Analogous. □

Theorem 5. Λ^λ*, under ordinal exponentiation, is a reverse-*ω*-point iff* λ *is any successor or* ω^**-point.*

Proof. Iff λ is successor Λ^λ is equal to $\Lambda^\alpha \cdot \Lambda$ for some α, and by preservation of technical elements the order ends with technical elements ordered as ω^*. Iff λ is an ω^*-point then by preservation of technical elements Λ^λ ends with technical elements ordered as ω^*. Iff λ is a non-ω^*-point limit then Λ^λ is the union $\Lambda^{\alpha<\lambda}$ which contains no greatest power of Λ and hence neither contains a greatest element nor ends with technical elements ordered as ω^*.

□

Definition of cardinal successor needs to be slightly adjusted as well.

Definition 2.8 (Cardinal successor of ω-almost-wellordering)**.** Cardinal successor of κ where κ is an ω-almost-wellordering is the least ω-almost-wellordering λ (up to isomorphism) such that there is no injection f : $\lambda \to \kappa$.

With this definition nothing prevents us from iterating cardinal successor over ω-almost-wellorderings.

Cofinality of ω-almost-wellorderings is especially well-behaved. If we exclude the impact of "technical elements" on lengths of cofinal sub-orders, cofinalities would behave exactly as expected.

Definition 2.9 (Cofinality of ω-almost-wellorderings)**.** $cof(\lambda)$ for ω-almost-wellorderings λ is defined as the least cardinality of sub-collections A $\subset \lambda$ such that there is no non-technical elements b $\in \lambda$ greater than all non-technical elements a $\in$ A

Theorem 6. *Cofinality of cardinal successor of* Λ *(*Λ^+ *for short) is* Λ^+*.*

Proof. Consider cof(Λ^+) $< \Lambda^+$, then Λ^+ is at most a union of Λ-many initial segments, taken at elements of the cofinal sequence, each sized at most Λ. This establishes a bijection between Λ^+ and at most $\Lambda \cdot 2$, however $\Lambda \cdot 2$ is equinumerous to Λ, contradiction. □

Theorem 7. *If* κ *is a cardinal successor of an* ω*-almost-wellordering* λ *then cof(*κ*) =* κ*.*

Proof. Analogous. □

[We immediately infer that if cof(κ) $< \kappa$, then κ is not a successor of any λ, in other words, the familiar property of successor and singular cardinals is preserved]

We may consider inaccessible ω-almost-wellorderings, Mahlo ω-almost-wellorderings, etc. Nothing prevents us from iterating even the "α-sort successor" operation on ω-almost-wellorderings. It behaves exactly the same way it does on well-orders. It doesn't interfere with any of the structural properties of orders involved as all it does is introduce an ontological shift to a higher type of collections, each iteration of α-sort successor acts as jump to a much muc much larger regular limit ω-almost-wellordering. Going through Λ, $\omega_{\dagger\Lambda+1}$, $\omega_{\dagger\Lambda+2}$, $\omega_{\dagger\Lambda^+}$, etc... We eventually get stuck once again and reach a special Burali-Forti limit.

Definition 2.10 (($\omega^* \cdot 2$)-limit)**.** An ω-almost-wellordering λ is ($\omega^* \cdot 2$)-limit iff λ contains an end segment of technical elements of order type $\omega^* \cdot 2$.

Although, it is consistent to assume that no ω-almost-wellordering λ is ($\omega^* \cdot 2$)-limit, such property of λ is less structurally violating than violation of ω-almost-wellorderability, so, due to the principle of *minimal cumulative structural violation* it is better to assume the existence of ω-almost-wellordered ($\omega^* \cdot 2$)-limits.

Remark. *Analogously defined ω-almost-wellordered $(\omega^* \cdot n)$-limits, for all finite ordinals n also exist under similar reasoning.*

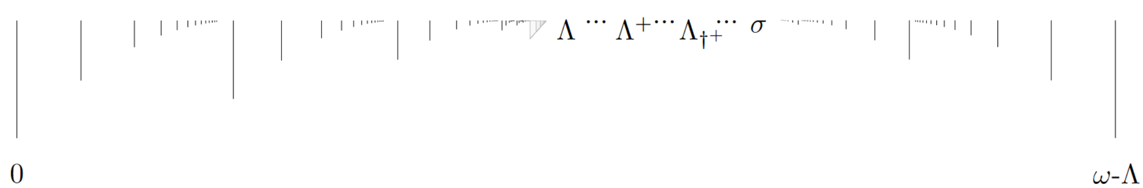
It is notable that the supremum of least $(\omega^* \cdot n)$-limits, for finite ordinals n, need not fail to be ω-almost-wellorderable. In fact, the supremum may not even be an ω^*-point, and, under axioms which cohere with the principle of *minimal cumulative structural violation*, it shouldn't be. It is also possible to consider $(\omega^* \cdot n)$-limits with large cardinal or reflective properties which, under the same principle, should be much larger than their non-$(\omega^* \cdot n)$-limit counterparts.

Finally, once more, we have to drop a structural property of the orders involved in our hierarchy. Dropping liearity of orders or consistency of the background logic once again proves to be too drastic of a change and we instead let go of well-orderability just slightly more drastically than before. We let go of ω-almost-wellorderability, and instead require our orders to be just ω^2-almost-wellordered.

2.3 ω^2-almost-wellorderings

Define an order to be ω^2-almost-wellordered iff it is linear and contains no strictly decreasing chains of length ω^3

Motivation for such a definition is now more clear. The hierarchy continues smoothly and is essentially the same as before



Definition 2.11 (Λ)**.** ω-Λ is the supremum of all ω-almost-wellorderings regardless of their collection types.

In similar spirit to Λ, ω-Λ has no ω-almost-wellordered initial segment that is equal to exactly the union of all ω-almost-wellorderings. Technical elements at the end of ω-Λ are ordered as reverse ω^2, the next additively indecomposable ordinal after ω. This greater degree of ill-foundedness allows one to minimally violate ω-almost-wellorederability and solve the relevant version of Burali-Forti paradox. ω-Λ, in many ways, is similar to Λ and plays a similar role. ω-Λ marks the point, past which counting numbers contain two different types of reverse limit points, reverse-ω-points and reverse-ω^2-points, the latter occurring much more sparsely.

Definition 2.12 (the 2 fundamental types of reverse limit points of ω^2-almost-wellorderings)**.**

1. ω^*-point: λ such that $\exists$ technical b $\in \lambda$ not bounded by any non-technical a $\in \lambda$ and the order type of technical elements c $\geq$ b is $\geq \omega$
2. ω^{2^*}-point: λ such that $\exists$ technical b $\in \lambda$ not bounded by any non-technical a $\in \lambda$ and the order type of technical elements c $\geq$ b is $\geq \omega^2$

Definitions of ordinal successor, sum, product and exponentiation, cardinal successor, cofinality and the principle of preservation of technical elements for ω^2-almost-wellorderings are exactly analogous to the according definitions for ω-almost-wellorderings modulo trivial adjustments.

Analogously to $(\omega^* \cdot n)$-limits, $(\omega^{2^*} \cdot n)$-limits, for each finite ordinal n, should exist within the hierarchy of ω^2-almost-wellorderings ordered under order-preserving embeddings.

Instead of ascending one step higher and defining ω^3-almost-wellorderings and ω^2-Λ we shall conceptually zoom out and consider general α-almost-wellorderings for well-orderings, and λ-almost-wellorderings α.

2.4 General λ-almost-wellorderings

The general pattern of the hierarchy of λ-almost-wellorderings is somewhat straightforward for λ which are limit well-orderings, howver for α-almost-wellordered λ, $\alpha \geq \omega$ the hierarchy is highly nontrivial. One may consider "large cardinal" variants of such orderings.

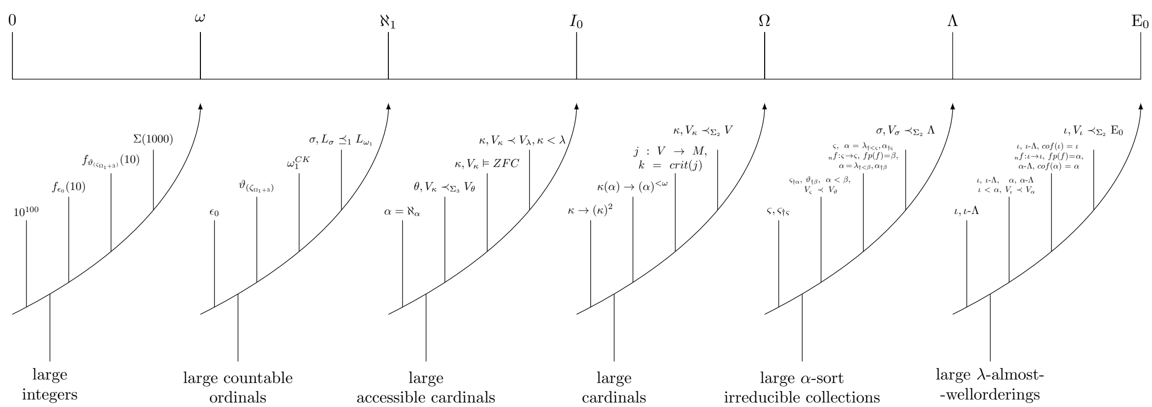
Definition 2.13 (λ-almost-fixed point)**.** λ is λ-almost-fixed point iff λ is a fixed point of the function $\alpha \mapsto \alpha$-Λ

Definition 2.14 (λ-almost-hyperfixed point)**.** λ is λ-almost-fixed point iff λ is a fixed point of the function $\alpha \mapsto \alpha$-th fp of ($\alpha \mapsto \alpha$-Λ)

Definition 2.15 (λ-almost-inaccessible)**.** κ is λ-almost-inaccessible iff κ is a κ-almost-wellordering and $cof(\kappa) = \kappa$

Definition 2.16 (λ-almost-Mahlo)**.** κ is λ-almost-Mahlo iff κ is a κ-almost-wellordering, $cof(\kappa) = \kappa$ any normal function f : $\kappa \to \kappa$ has a λ-almost-inaccessible fixed point.

It is clear that the intermediate hierarchies of reflective properties such as admissibility degrees, large cardinal hierarchies, concreteness degrees and higher are layered to truly colossal degrees at the level of λ-almost-large cardinals. Due to this reason, it is nearly impossible to visualize the layers to appropriate level of detail. The fine structure of the layering is a possible direction for future research.



Definition 2.17 (E_0)**.** E_0 is the supremum of all hereditarily ${}_{\leq}\kappa$-almost-well-orderings. Equivalently, E_0 is the least ∞-almost counting number or the least counting number ξ such that $\xi \in \xi$.

It is possible to continue the hierarchy of λ-almost-wellorderings past E_0, however, to truly visualize the scope of all counting numbers, it is necessary to take the bird's-eye view and consider much stronger and much more general notions and extensions.

3 The notion of Burali-Forti solutions

Burali-Forti solution is a technical semi-formal term which embodies the global approach to generalizing ordinals to greater counting numbers as shown in the paper.

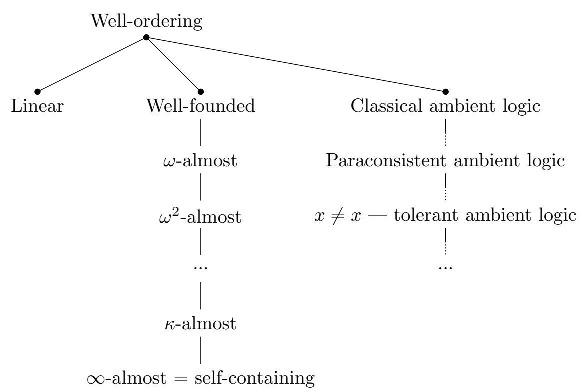
Definition 3.1 (Counting number)**.** Counting number is a term which refers to generalizations of ordinals to greater linear orders such that shorter initial segments of the number line have logically stronger structural properties.

Definition 3.2 (BFS)**.** Abbreviation BFS stands for Burali-Forti solution. BFS is a system of limit counting numbers which are suprema of structurally equally well-behaved counting numbers with respect to a particular structural feature (e.g., cardinality, admissibility degree, concreteness degree).

Definition 3.3 (BFL)**.** Abbreviation BFS stands for Burali-Forti limit. BFL is counting number which is the supremum of a particular collection of BFS.

Definition 3.4 (Good system of BFS)**.** A good system G is an α-sized collection of BFS indexed by β, $|\beta| = \alpha$ such that BFS_γ is less structurally violating than BFS_δ iff $\gamma < \delta$, and for any two sub-collections G_A, G_B of the collection G, $\text{G}_A \not\Leftrightarrow \text{G}_B$ modulo ambient logic of G.

Definition 3.5 (Supremum of a good system)**.** $sup(G) = \alpha$ iff α is the least counting number such that $\text{BFS}_\beta(\alpha)$ for all $\text{BFS}_\beta \in \text{G}$



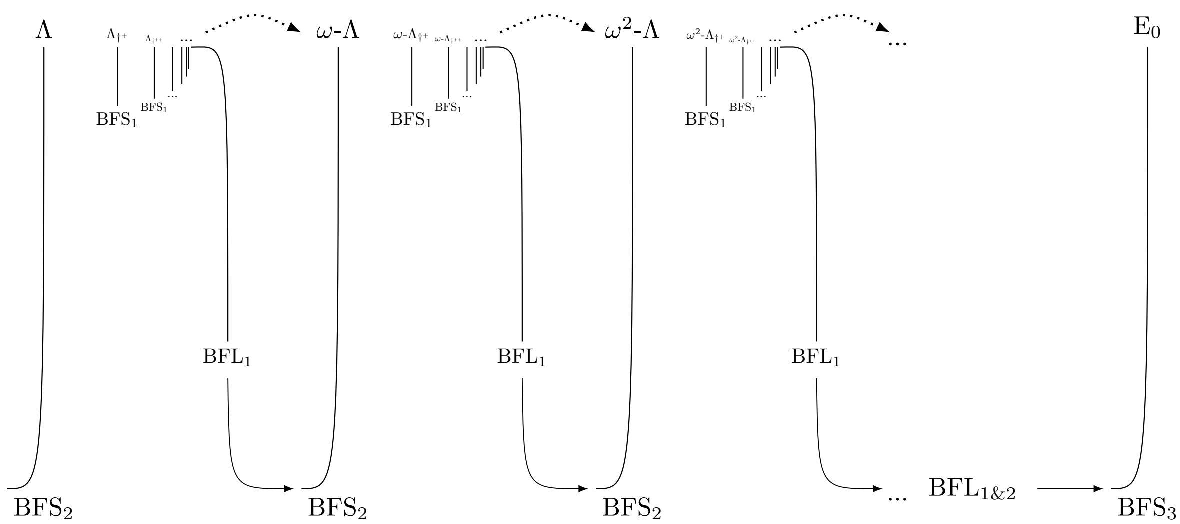
The concept of a well-ordering can be seen as a poset of its characterizing properties ordered by structural freedom with the greatest element being the property of being well-ordered. A sub-order of the poset is shown in the diagram.[‡]

If the concept of well-ordering is seen through the lens of being a special case of the concept of counting numbers, the property of linearity is a minimal element as linearity of the number line is indispensable from the conception of counting.

The purpose of good systems is to linearize sub-orders of the poset. Linearity provides the basis of the progression of the counting number line in which increase in magnitudes correlates with the increase in structural freedom. Good systems which linearize the poset completely shall be notated as **great systems**.

Notably, ambient logics in which the concept is considered can also be seen as a characterizing property. It is easy to present philosophical arguments that in proper linearizations of the poset these properties appear relatively deep as counting numbers which are greater than themselves or are not self-identical are highly structurally free.

[‡]The poset is need not form a set



As shown in the diagram, α-sort irreducibility, λ-almost-wellorderability and ∞-almost-wellorderability form a good system of cardinality 3.

Truly colossal counting numbers emerge if we consider those which satisfy reflective or large cardinal properties in the terminology of BFS and good systems. Such counting numbers celebrate their leadership over all counting numbers described so in the previous passages, even those which are too large not to satisfy bizarre properties such as being amenable exclusively to $\kappa_{\dagger^+}$-valued paraconsistent ambient logics or even satisfying $x \neq x$.

3.1 Formalism-inaccessibility

Definition 3.6 (The least Formalism-inaccessible number)**.** F_0 is the least counting number ξ such that for any good system $|G| < \xi$, $sup(G) < \xi$.

Definition 3.7 (α-th Formalism-inaccessible number)**.** F_α, for counting numbers α is α-th counting number ξ such that for any good system $|G| < \xi$, $sup(G) < \xi$.

Formalism-2-inaccessible numbers are precisely the Formalism-inaccessible limits of Formalism-inaccessibles.

Definition 3.8 (α-th Formalism-Mahlo number)**.** F_α, for counting numbers α is α-th counting number ξ such that ξ is Formalism-inaccessible and any normal function f : $\xi \to \xi$ has a Formalism-inaccessible fixed point.

Nothing prevents the translation of some greater large cardinal and reflective properties into properties of good systems, as well as translation of properties between Mahloness and inaccessibility and below, however, such high-level constructions approach an important limit.

Having considered reflective properties of good systems we are finally forced to let go of any remnants of the assumption that all counting numbers are amenable to formal ontology. While formal ontology of numbers such as "the least Formalism-2-Mahlo" exists but is completely inaccessible to formal investigation, greater orders can no longer can be given neither purely formal structural ontology nor can be essentially formal-structural in any significant way by a direct consequence of their colossal size. We are finally forced into the realm of unformalizable subtleties and ethereal aspects of metaphysics.

4 The least completely unformalizable counting number

Definition 4.1. $\mathscr{K}$ is the supremum of a **great system**,

equivalently, $\mathscr{K}$ is the least unformalizable counting number.

This counting number marks the end of formal structural approach to counting numbers, since all formal structures are of cardinality $< \mathscr{K}$. $\mathscr{K}$ has deep philosophical and, in particular, metaphysical implications. Any domain of discourse, which $\mathscr{K}$ is an element of must be strictly informal logical in nature, or in other words, **unformalizable**. Unformalizability is to be contrasted with insufficient rigorization, existence of obstacles for sufficient rigorization or impossibility of recursively enumerable axiomatization. A concept is **unforamlizable** if it cannot be rigorized even in principle.

$\mathscr{K}$ plays the role of a concrete example of an abstract object so vast that it is cannot be analyzed via formal methods, yet still can be the target of a mental state and analyzed via methods of informal logic. This number marks a hierarchy of counting numbers much more vast then the mere hierarchy of formalizable counting numbers. Nothing prevents us from forming mental states whose targets are $\mathscr{K} + 1$, $\mathscr{K} \cdot 2$, $\mathscr{K}_{\dagger +}$ and so on, except now, we are not limited by formalizable operations.

Because of the concievability of such extensions,

even $\mathscr{K}$ is unimaginably smaller than the magnitude of absolute totality.

5 The magnitude of Absolute Totality

Definition 5.1. $\mathscr{A}$ is the supremum of *absolutely* all counting numbers,

equivalently, $\mathscr{A}$ is the *absolutely* greatest counting number

Immediately from the definition, it is inferred that $\mathscr{A}$ is greater than all counting numbers greater than it, so $\mathscr{A}$ is closed under the direct conception of greater counting numbers. It is extraordinarily difficult to defend the position that there are ways in which greater counting numbers are conceivable, however such position exists and shall be referred to as “trans-absolutism”.

Being a trans-absolutist is equivalent to defending the position that it is possible to conceive of agents which hold greater powers than even the absolutist interpretation of Omnipotence.