Transfinite FGH

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1 Definition

For some s, let $\text{cof}(s)$ denote its cofinality, i.e. the minimum cardinality $|S|$ of some set S with supremum s. This serves as an indicator of the number's type:

1. If $s = 0$, then $\text{cof}(s) = 0$, since we can construct the empty set, which has supremum 0.
2. If $s = s' + 1$ for some s', then $\text{cof}(s) = 1$, since we can construct the set $\{s'\}$
3. If s is a countable limit ordinal, then $\text{cof}(s) = \omega$.
4. etc.

Also, along with cofinality, we also need **fundamental sequences** to define our transfinite FGH. For some s, we define the fundamental sequence of s, whose nth element is denoted by $s[n]$.

The fundamental sequence for some s is a sequence with length $\text{cof}(s)$, and its limit $\lim_{n\to\text{cof}(s)} s[n] = s$. So, the fundamental sequence for 0 is obviously the empty sequence, since 0 has cofinality 0 and the empty sequence has length 0.

Note that, if $\text{cof}(s) = 1$, then $s[0]$ is the predecessor of s.

The exact fundamental sequences we use are somewhat important, but I will ignore a definition, since many different systems exist.

Definition: I define a function $f_s^m(n)$ like so:

1. If $m = 0$, then $f_s^m(n) = n$.
2. If $m = 1$:
   (a) If $s = 0$, then $f_s^m(n) = n + 1$.
   (b) If $\text{cof}(s) = 1$, then $f_s^m(n) = f_{s[0]}^n(n)$
   (c) Else:
      i. If $n < \text{cof}(s)$, then $f_s^m(n) = f_{s[n]}^m(n)$.

ii. Else, $f_s^m(n) = \lim_{k \to s} f_k^m(n)$.

3. Else:

   (a) If $cof(m) = 1$, then define $g_s^m(n)$ to be the least number $\geq f_s^m(n)$ so that $f_s^1(g_s^m(n)) \neq g_s^m(n)$. Then, $f_s^m(n) = f_s^1(g_s^{m[0]}(n))$.

   (b) Else, $f_s^m(n) = \lim_{k \to m} f_s^k(n)$

When restricted to finite values of n, this is the same as the **fast-growing hierarchy** (FGH). However, this provides a transfinite extension.

Let us define $\Lambda(n) = f_{S_n}^1(n)$, where S_n is defined recursively like so:

1. If $k = 0$, then $S_k = 0$

2. Else, $S_k = f_{S_{k[0]}}(\omega)$.

Then, $\Lambda(n)$ grows approximately at the same rate as $f_{\Gamma_0}(n)$ with respect to the finite fast-growing fast hierarchy.