**The general plan**

1. Our starting point is the set of ordinals below $\varepsilon_0$.
2. Extending this into a continuum of ordinals and defining addition, multiplication and omega powers on them. These will be called *FOATS* ("fractional ordinals at ten").
3. Defining continuous fundamental sequences for regular ordinals. Note: *FOATS* do *not* have fundamental sequences.
4. Using the above to define a backwards-compatible version of the letters up to *P*, and to define *Q*.
5. Outlining a companion array notation to facilitate ease of use.

**Fractional Ordinals At Ten (*FOATs*)**

Let $F = \varepsilon_0 \times [0,1)$. An element of *F* be called a "Fractional Ordinal At Ten" (*FOAT*).

Let $\alpha \in \varepsilon_0$ , $(\beta,t) \in F$ and $x \in [0,\infty)$. Also, let $I = \lfloor x \rfloor$ and $F = x - I$ . We then define the following operations:

(1) Converting a real number into a *FOAT*:

$RTF(x)=(I,F)$ (result is a *FOAT*)

(2) Addition of an ordinary ordinal and a *FOAT:*

$\alpha+(\beta,t) = (\alpha+\beta,t)$ (result is a *FOAT*)

(3) Multipliation of an omega power and a nonnegative real number:

(*i*) $\omega^0 \cdot x = 1 \cdot x = RTF(x)$ (result is a *FOAT*)

(*ii*) $\omega^\alpha \cdot x = \omega^\alpha \cdot I + FS(\omega^\alpha, 10F)$ (result is a *FOAT*)

(*FS*($\alpha$,*x*) basically means "the *x*-th member of the fundamental sequence of $\alpha$" and it will be defined in the next section)

(4) Omega powers to *FOATs*:

$\omega^{(\beta,t)} = \omega^\beta \cdot 10^t$ (result is a *FOAT*)

(5) Converting a *FOAT* with no fractional part into an ordinal:

$FTO((\beta,0)) = \beta$ (result is **an ordinal**)

**Continuous Fundamental Sequences**

Let $E = \varepsilon_0 \cup \{\varepsilon_0\}$. Then we define a function $FS : E \times [0,\infty) \rightarrow F$ as follows:

Let $\alpha \in E$ and $x \in [0,\infty)$. As before, let $I = \lfloor x \rfloor$ and $F = x - I$ . We then define:

(1) If $\exists \beta,\gamma$: $\alpha = \beta+\omega^\gamma \wedge \beta \geq \omega^\gamma$ then $FS(\alpha,x) = \beta+FS(\omega^\gamma,x)$

(2) $FS(\omega^{\alpha+1},x) = \omega^\alpha \cdot x$

(3) If $\exists \beta<\varepsilon_0 : \alpha=\omega \cdot \beta$ and $x \geq 1$ then $FS(\omega^\alpha,x) = \omega^{FS(\alpha,x)}$

(4) If $\exists \beta<\varepsilon_0 : \alpha=\omega \cdot \beta$ and $x<1$ then $FS(\omega^\alpha,x) = \omega^{FTO(FS(\alpha,1))} \cdot x$

(5) If $0<x<1$ then $FS(\varepsilon_0,x) = RTF(10^x)$

(6) If $x \geq 1$ then $FS(\varepsilon_0,x) = \omega^{FS(\varepsilon_0,\, x-1)}$

**Defining Letter Notation Up to Q**

Now we are ready to define our notation:

(1) Valid forms are one of the following:

($i$) $[\alpha]x$ where $\alpha \in E$ and $x \in [0,\infty)$

($ii$) $[(\beta,t)]10$ where $(\beta,t) \in F$

(2) $[1]x = 10^x$

(3) For $x \le 1$: $[\alpha]x = 10^x$

(4) For $x > 1$: $[\alpha+1]x = [\alpha][\alpha+1]x-1$

(5) If $\exists\gamma : \alpha = \omega\cdot\gamma$ and $x<2$ then $[\alpha]x = [FTO(FS(\alpha,2)+1)]x$

(6) If $\exists\gamma : \alpha = \omega\cdot\gamma$ and $x \ge 2$ then $[\alpha]x = [FS(\alpha,x)]10$

(7) $[(\beta,t)]10 = [\beta+1](2\cdot 5^t)$

(8) $Ex = [1]x$ , $Fx = [2]x$ , $Gx = [3]x$ , $Hx = [4]x$ , $Jx = [\omega]x$ , $Kx = [\omega+1]x$ , $Lx = [\omega+2]x$ ,

$Mx = [\omega\cdot 2]x$ , $Nx = [\omega^2]x$ , $Px = [\omega^\omega]x$ , $Qx = [\varepsilon_0]x$

Note that if we write:

(1) $[\omega^n\cdot a_n+\omega^{n-1}\cdot a_{n-1}+...+\omega\cdot a_1+a_0]$ as $[a_n , a_{n-1} , ... ,a_1 , a_0]$

(2) $[(\omega^n\cdot a_n+\omega^{n-1}\cdot a_{n-1}+...+\omega\cdot a_1+a_0 , t)]$ as $[a_n , a_{n-1} , ... ,a_1 , a_0+t]$

(3) $[(\omega^n\cdot a_n+\omega^{n-1}\cdot a_{n-1}+...+\omega^k\cdot x)]$ as $[a_n , a_{n-1} , ... ,a_{k+1} , x , 0 , ... , 0]$

Then the above definition is backwards-compatible with the previously published array-based definitions.

**Associated Array Notation - Extending it to nested arrays**

To facilicate calculation and notation, we can write ordinals and *FOATS* as nested arrays:

(1) The number $x$ is represented by the array '$x$'.

(2) If $A$ represents $\alpha$ and $B$ represents $\beta$ then:

($i$) if $\omega^{\omega^\alpha} > \beta$ then '$x$ ($A$) $B$' represents $\omega^{\omega^A}\cdot x+B$

($ii$) if $\omega^{\omega^\alpha} < \beta$ then '$x$ ($A$) $B$' represents $\omega^{\omega^A}\cdot B\cdot x$

(3) If $A$ represents $\alpha$ then '1 $(A)_x$ 0' represents $\omega^{\omega^{A\cdot x}}$

This notation doesn't stand on it's own, but it has the advantage of mimicing the behavior of Bird/Bowers nested arrays which many of you are comfortable with.

**Example**

Let's calculate Q2.021, first by using the concise *FOAT*-based definition and then repeating the calculation more intuitively:

| | | |
|---|---|---|
| Q2.021 = | $[\varepsilon_0]2.021$ | |
| = | $[FS(\varepsilon_0,2.021)]10$ | (Expansion Rule 6) |
| = | $[\omega^{FS(\varepsilon_0\,,\,1.021)}]10$ | (FS Rule 6) |

= $[\omega^{\omega^{FS(\varepsilon_0,0.021)}}]10$ (FS Rule 6)

= $[\omega^{\omega^{RTF(10^{0.021})}}]10$ (FS Rule 5)

= $[\omega^{\omega^{RTF(1.049542428652322)}}]10$

= $[\omega^{\omega^{(1,0.049542428652322\ )}}]10$ (Convert Number to *FOAT*)

= $[\omega^{\omega\cdot 10^{0.049542428652322}}]10$ (*FOAT* power rule)

= $[\omega^{\omega\cdot 1.1208369216037}]10$

= $[\omega^{\omega+FS(\omega,1.208369216037)}]10$ (*FOAT* multiplication rule 2)

= $[\omega^{\omega+1\cdot 1.208369216037}]10$ (FS Rule 2)

= $[\omega^{\omega+RTF(1.208369216037)}]10$ (*FOAT* multiplication rule 1)

= $[\omega^{\omega+(1,0.208369216037)}]10$ (Convert Number to *FOAT*)

= $[\omega^{(\omega+1,0.208369216037\ )}]10$ (*FOAT* addition rule)

= $[\omega^{\omega+1}\cdot 10^{0.208369216037}]10$ (*FOAT* power rule)

= $[\omega^{\omega+1}\cdot 1.61573158948]10$

= $[\omega^{\omega+1}+FS(\omega^{\omega+1},6.1573158948)]10$ (*FOAT* multiplication rule 2)

= $[\omega^{\omega+1}+\omega^{\omega}\cdot 6.1573158948]10$ (FS Rule 2)

= $[\omega^{\omega+1}+(\omega^{\omega}\cdot 6+FS(\omega^{\omega},1.573158948))]10$ (*FOAT* multiplication rule 2)

= $[\omega^{\omega+1}+(\omega^{\omega}\cdot 6+\omega^{FS(\omega,1.573158948)})]10$ (FS Rule 3)

= $[\omega^{\omega+1}+(\omega^{\omega}\cdot 6+\omega^{1\cdot 1.573158948})]10$ (FS Rule 2)

= $[\omega^{\omega+1}+(\omega^{\omega}\cdot 6+\omega^{RTF(1.573158948)})]10$ (*FOAT* multiplication rule 1)

= $[\omega^{\omega+1}+(\omega^{\omega}\cdot 6+\omega^{(1,0.573158948)})]10$ (Convert Number to *FOAT*)

= $[\omega^{\omega+1}+(\omega^{\omega}\cdot 6+\omega\cdot 10^{0.573158948})]10$ (*FOAT* power rule)

= $[\omega^{\omega+1}+(\omega^{\omega}\cdot 6+\omega\cdot 3.7424753)]10$

= $[\omega^{\omega+1}+(\omega^{\omega}\cdot 6+(\omega\cdot 3+FS(\omega,7.424753)))]10$ (*FOAT* multiplication rule 2)

= $[\omega^{\omega+1}+(\omega^{\omega}\cdot 6+(\omega\cdot 3+1\cdot 7.424753))]10$ (FS Rule 2)

= $[\omega^{\omega+1}+(\omega^{\omega}\cdot 6+(\omega\cdot 3+RTF(7.424753)))]10$ (*FOAT* multiplication rule 1)

= $[\omega^{\omega+1}+(\omega^{\omega}\cdot 6+(\omega\cdot 3+(7,0.424753)))]10$ (Convert Number to *FOAT*)

= $[\omega^{\omega+1}+(\omega^{\omega}\cdot 6+(\omega\cdot 3+7,0.424753))]10$ (*FOAT* addition rule)

= $[\omega^{\omega+1}+(\omega^{\omega}\cdot 6+\omega\cdot 3+7,0.424753)]10$ (*FOAT* addition rule)

= $[(\omega^{\omega+1}+\omega^{\omega}\cdot 6+\omega\cdot 3+7,0.424753)]10$ (*FOAT* addition rule)

= $[\omega^{\omega+1}+\omega^{\omega}\cdot 6+\omega\cdot 3+8]2\cdot 5^{0.424753}$ (Expansion Rule 7)

= $[\omega^{\omega+1}+\omega^{\omega}\cdot 6+\omega\cdot 3+8]3.96205$

= $[\omega^{\omega+1}+\omega^{\omega}\cdot 6+\omega\cdot 3+7][\omega^{\omega+1}+\omega^{\omega}\cdot 6+\omega\cdot 3+8]2.96205$ (Expansion Rule 4)

= $[\omega^{\omega+1}+\omega^{\omega}\cdot 6+\omega\cdot 3+7]_2[\omega^{\omega+1}+\omega^{\omega}\cdot 6+\omega\cdot 3+8]1.96205$ (Expansion Rule 4)

= $[\omega^{\omega+1}+\omega^{\omega}\cdot 6+\omega\cdot 3+7]_3[\omega^{\omega+1}+\omega^{\omega}\cdot 6+\omega\cdot 3+8]0.96205$ (Expansion Rule 4)

= $[\omega^{\omega+1}+\omega^{\omega}\cdot 6+\omega\cdot 3+7]_3 10^{0.96205}$ (Expansion Rule 3)

= $[\omega^{\omega+1}+\omega^{\omega}\cdot 6+\omega\cdot 3+7]_3 9.163$

= ...

With some practice, you could do the above calculation much more quickly:

| | | |
|---|---|---|
| Q2.021 = | $[\varepsilon_0]2.021$ | |
| = | $[\omega^{\omega^{10^{0.021}}}]10$ | (power tower of $int(2.021)$ $\omega$'s topped by $10^{frac(2.021)}$) |
| = | $[\omega^{\omega^{1.049542428652322}}]10$ | |
| = | $[\omega^{\omega+1.208369216037}]10$ | ($10^{1.0495...}$ = 11.208... = $10^1$+1.208... → $\omega^1$+1.208...) |
| = | $[\omega^{\omega+1}\cdot 1.61573158948]10$ | ($10^{0.2083...}$ = 1.6157...) |
| = | $[\omega^{\omega+1}+\omega^{\omega}\cdot 6+\omega^{1.573158948}]10$ | (1.61573... → $\omega^{\omega+1}\cdot 1+\omega^{\omega}\cdot 6+FS(\omega^{\omega},1.573...)$) |
| = | $[\omega^{\omega+1}+\omega^{\omega}\cdot 6+\omega\cdot 3+7.424753]10$ | ($10^{1.5731...}$ = 37.424... = $3\cdot 10^1$+7.424... → $\omega^1\cdot 3$+7.424...) |
| = | $[\omega^{\omega+1}+\omega^{\omega}\cdot 6+\omega\cdot 3+8]2\cdot 5^{0.424753}$ | |
| = | $[\omega^{\omega+1}+\omega^{\omega}\cdot 6+\omega\cdot 3+8]3.96205$ | |
| = | $[\omega^{\omega+1}+\omega^{\omega}\cdot 6+\omega\cdot 3+7]_3 10^{0.96205}$ | (in general $[\alpha+1]x = [\alpha]_{int(x)}10^{frac(x)}$) |
| = | $[\omega^{\omega+1}+\omega^{\omega}\cdot 6+\omega\cdot 3+7]_3 9.163$ | |

Which can also be written in array notation as $[1,6\ (1)\ 3,7]_3 9.163$